Transferring oceanographic to acoustic uncertainty using polynomial chaos expansions

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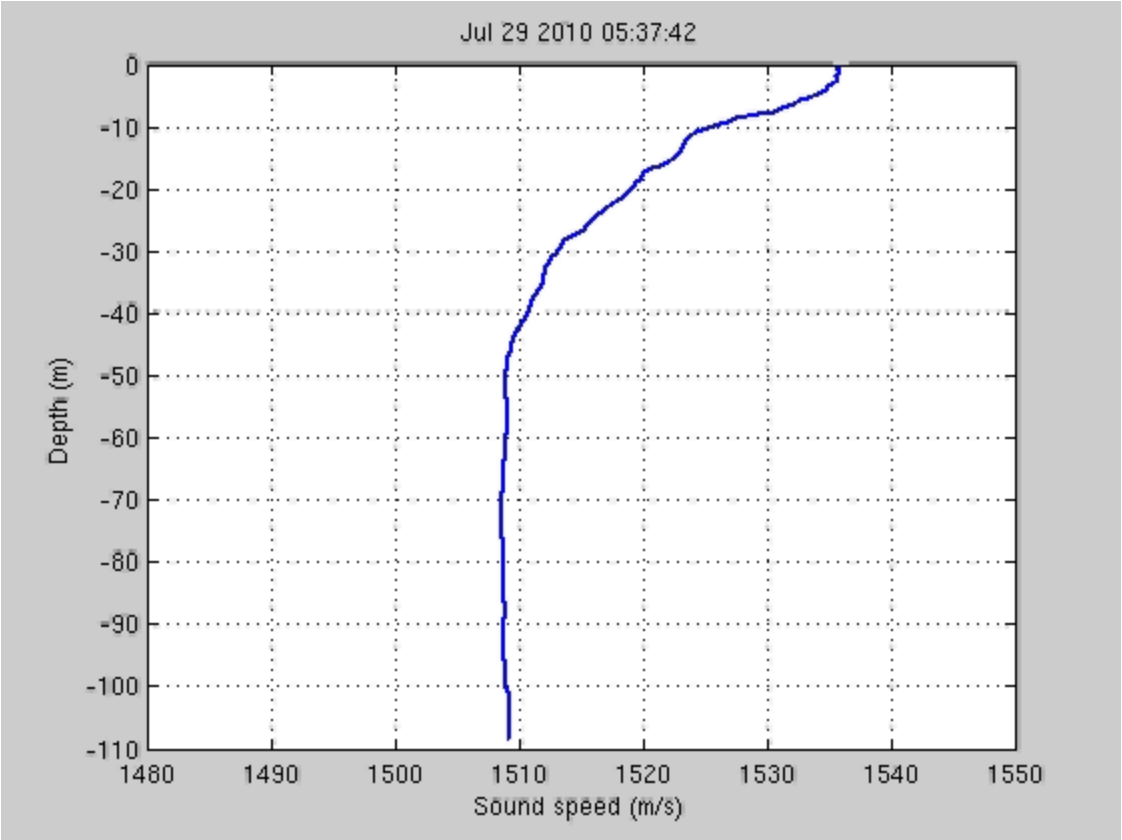
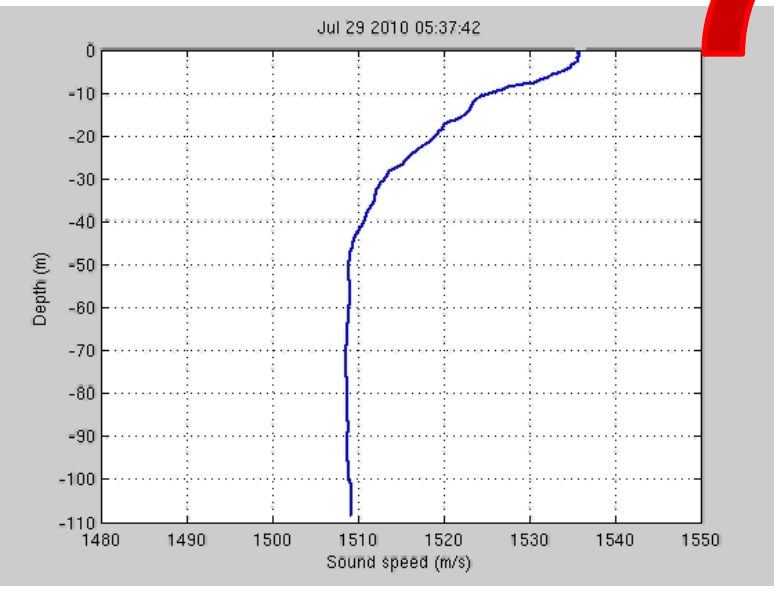
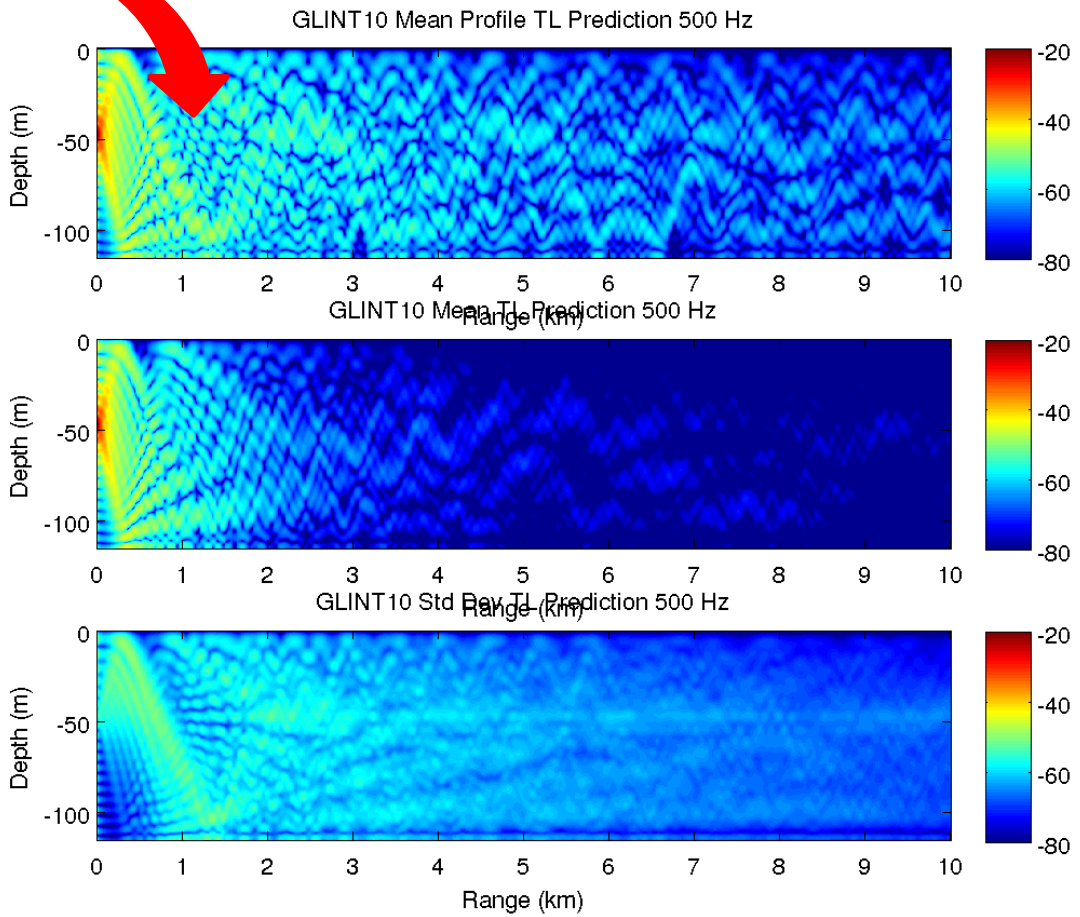
Motivation:
Environmental Variability Observed during GLINT10



Objectives

- **To develop forward acoustic modeling techniques which predict the probability density function (*pdf*) of the acoustic field for specified *pdfs* of the environmental uncertainty**





Technical Approach

Polynomial Chaos Expansions of Propagation through Uncertain Environments

- **Expansion of environmental uncertainty into uncorrelated EOFs with specified (uncorrelated Gaussian) *pdfs***
- **Expand the acoustic field in polynomials and products of polynomials in these uncorrelated GRVs (the Polynomial Chaos). Polynomials are orthogonal under the *pdf* weight which leads to a significant reduction in cross terms.**
- **Polynomial Chaos approach may be applied intrusively or non-intrusively**
  - **Intrusive formulation may provide more insight**
  - **Non-intrusive formulation may be more robust and is easier to apply to legacy codes**
- **Polynomial Chaos expansions are different than moment equations**
  - **When converged, any statistical property of the field may be calculated, including *pdf***

Introduction Polynomial Chaos into the Perturbed Coupled Mode Equation

- **Start with the 2-D 1-way coupled mode equation**

$$\frac{d\,a_n(x)}{dx}+\sum_{m=1}^{N}a_m(x)\int_{-\infty}^{0}\frac{\partial\phi_m(x,z)}{\partial x}\frac{\phi_n(x,z)}{\rho}dz\pm ik_n(x)a_n(x)=0$$

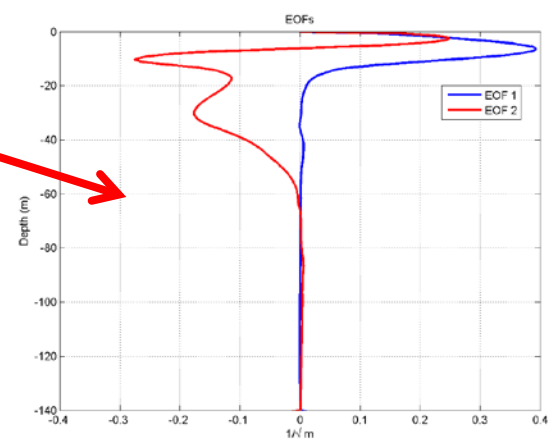
- **Factor the range and depth dependence of the environmental variability**

$$\frac{da_n(x)}{dx}+\sum_{m=1}^{N}a_m(x)A_{mn}\frac{d\tilde{\eta}(x)}{dx}\xi-i\left(k_n(x)+\Delta k_n\tilde{\eta}(x)\xi\right)a_n(x)=0$$

- **for ξ random use the perturbation approximations**

$$A_{mn}\simeq\frac{2\omega^2}{k_m^2-k_n^2}\left(\int_{-\infty}^{0}\frac{1}{r}\frac{\phi_m(z)\phi_n(z)}{c^2(z)\rho(z)}\frac{\delta c}{c}(z)dz\right)$$

$$\Delta k_n\simeq\left(-\frac{\omega^2}{k_n}\int_{-\infty}^{0}\frac{\phi_n^2(z)}{c^2(z)\rho(z)}\frac{\delta c}{c}(z)dz\right)$$



Using PC Expansions to Evaluate Adiabatic vs Non-Adiabatic Propagation Through Uncertainty

- Simple and powerful expressions for decorrelation of normal modes for adiabatic propagation in random waveguides exist
- Would like to evaluate for cases of interest the accuracy of the adiabatic assumption
  - When in a mode coupling environment is this a good approximation
  - For what modes
  - Interacting with what vertical and horizontal fluctuation scales
- The polynomial chaos expansion approach offers a tool for conducting such a study

Polynomial Chaos Expansion of Log Modal Amplitudes a Natural Choice for Studying Adiabaticity

- **Perturbation coupled mode equation**

$$\frac{da_n(x)}{dx} + \sum_{m=1}^{N} a_m(x) A_{mn} \frac{d\tilde{\eta}(x)}{dx}\xi - i\left(k_n(x) + \Delta k_n \tilde{\eta}(x)\xi\right) a_n(x) = 0$$

- **Polynomial Chaos Expansion**

$$a_n = a_n(0)\exp\left(ik_n^o x + \Theta_{n1}(x)\xi + \Theta_{n2}(x)\left(\xi^2 - 1\right) + \cdots + \Theta_{nN_{PC}}\Gamma_{N_{PC}}(\xi)\right)$$

- **In the Adiabatic Case**

$$\frac{da_n(x)}{dx} = i\left(k_n(x) + \Delta k_n \tilde{\eta}(x)\xi\right) a_n(x)$$

$$\Rightarrow a_n(x) = a_n(0)\exp\left(i\int_0^x \left(k_n(x') + \Delta k_n \tilde{\eta}(x)\xi\right) dx'\right)$$

$$\Rightarrow \Theta_{n1} = i\Delta k_n \tilde{\eta}(x)$$

$$\Theta_{n2} \cdots \Theta_{nN_{PC}} = 0$$

Polynomial Chaos Expansion of Log Modal Amplitudes a Natural Choice for Studying Adiabaticity

- **Perturbation coupled mode equation**

$$\frac{da_n(x)}{dx}+\sum_{m=1}^{N}a_m(x)A_{mn}\frac{d\tilde{\eta}(x)}{dx}\xi-i\left(k_n(x)+\Delta k_n\tilde{\eta}(x)\xi\right)a_n(x)=0$$

- **Polynomial Chaos Expansion**

$$a_n=a_n(0)\exp\left(ik_n^o x+\Theta_{n1}(x)\xi+\Theta_{n2}(x)\left(\xi^2-1\right)+\cdots+\Theta_{nN_{PC}}\Gamma_{N_{PC}}(\xi)\right)$$

- **But in the general Case**

$$\sum_{j}^{N_{PC}}\frac{d\Theta_{jn}}{dx}\Gamma_j(\xi_1)-i\Delta k_n\eta(x)\xi$$

$$=-\sum_{m=1}^{N}\frac{a_m(0)}{a_n(0)}\exp\left(i\left(k_m^o-k_n^o\right)x\right)\exp\left(\sum_{k}^{N_{PC}}\Theta_{km}(x)\Gamma_k(\xi)-\sum_{j}^{N_{PC}}\Theta_{jn}(x)\Gamma_j(\xi)\right)\left(A_{mn}\frac{d\eta}{dx}\xi\right)$$

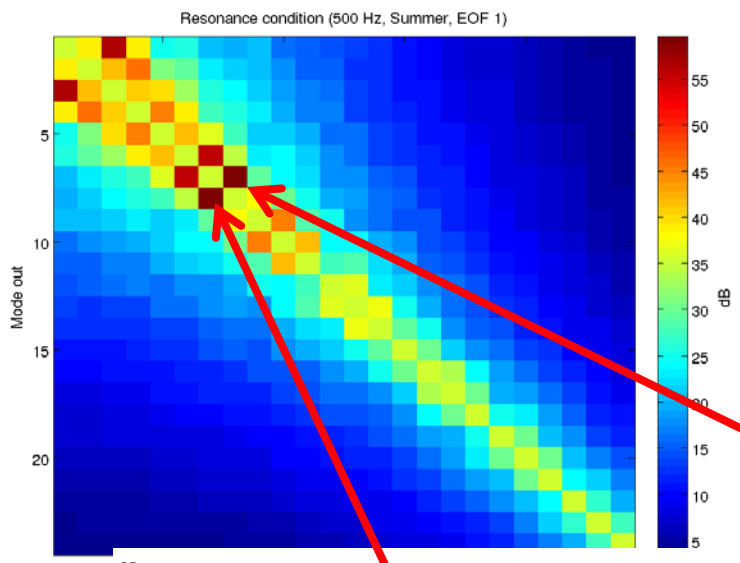
Horizontal and Vertical Resonances

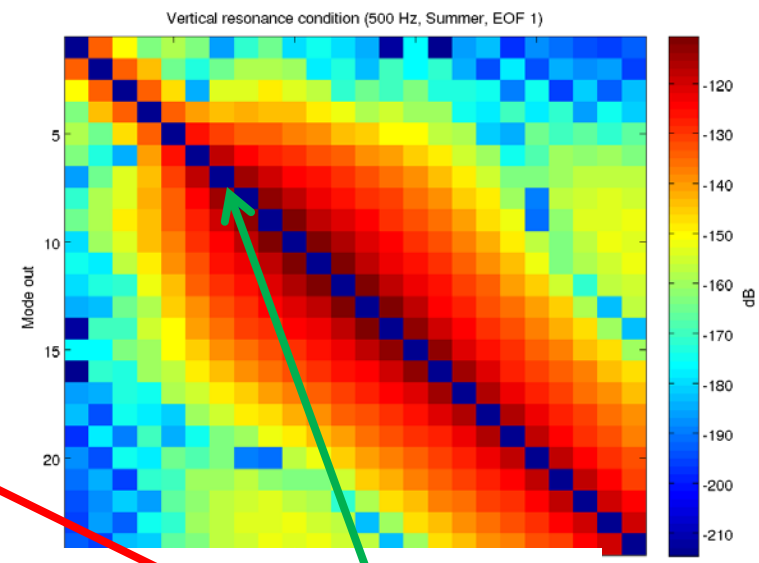
Horizontal Resonance

$$\int_0^r \exp\left(i\left(k_m^o - k_n^o\right)r'\right)\frac{d\eta(r')}{dr}dr'$$

Vertical Resonance

$$A_{mn} \simeq \frac{2\omega^2}{k_m^2 - k_n^2}\left(\int_{-\infty}^{0}\frac{1}{r}\frac{\phi_m(z)\phi_n(z)}{c^2(z)\rho(z)}\frac{\delta c}{c}(z)dz\right)$$



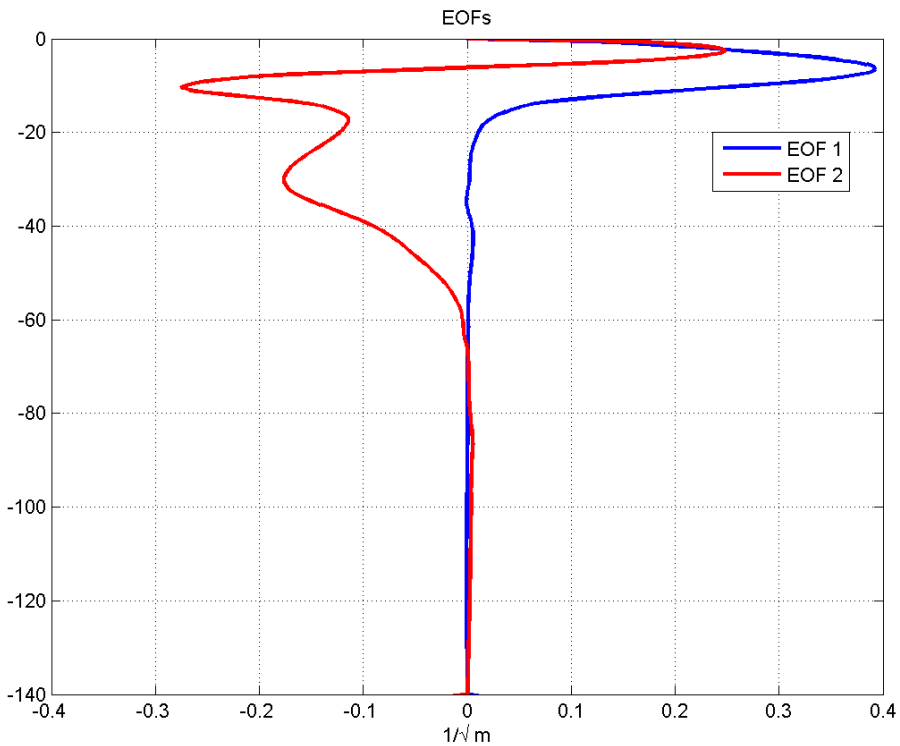


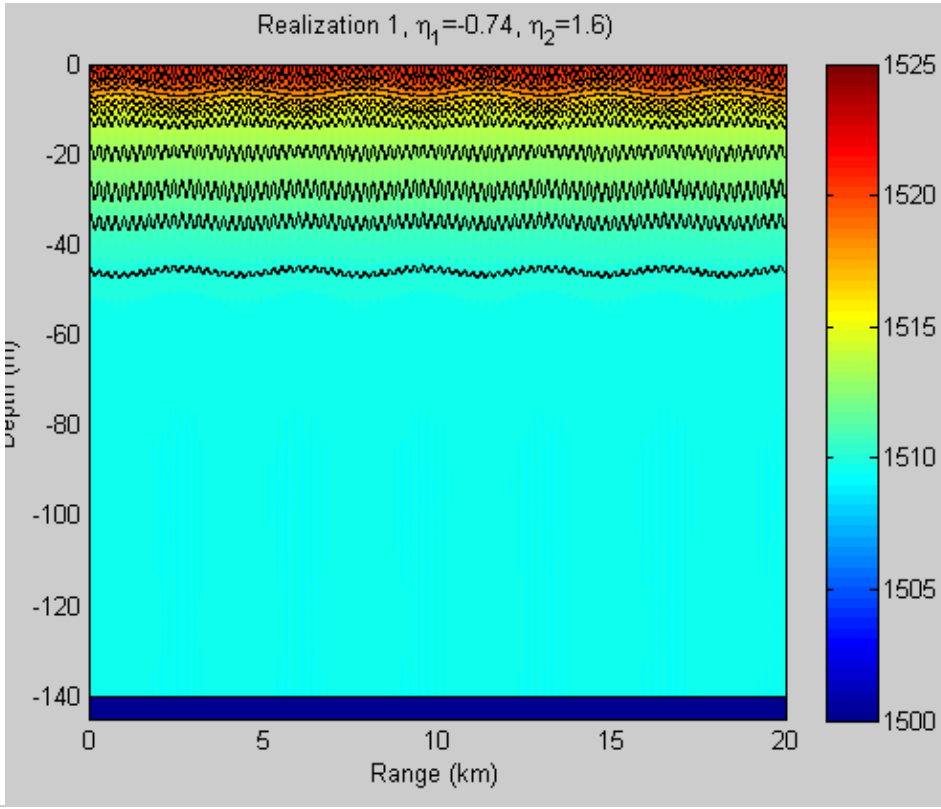
$$\sum_j^{N_{PC}}\frac{d\Theta_{jn}}{dx}\Gamma_j(\xi_1) - i\Delta k_n \eta\,(x)\xi$$

$$= -\sum_{m=1}^{N}\frac{a_m(0)}{a_n(0)}\exp\left(i\left(k_m^o - k_n^o\right)x\right)\exp\left(\sum_k^{N_{PC}}\Theta_{km}(x)\Gamma_k(\xi) - \sum_j^{N_{PC}}\Theta_{jn}(x)\Gamma_j(\xi)\right)\left(A_{mn}\frac{d\eta}{dx}\xi\right)$$

Example: 2 DOF Random Environment

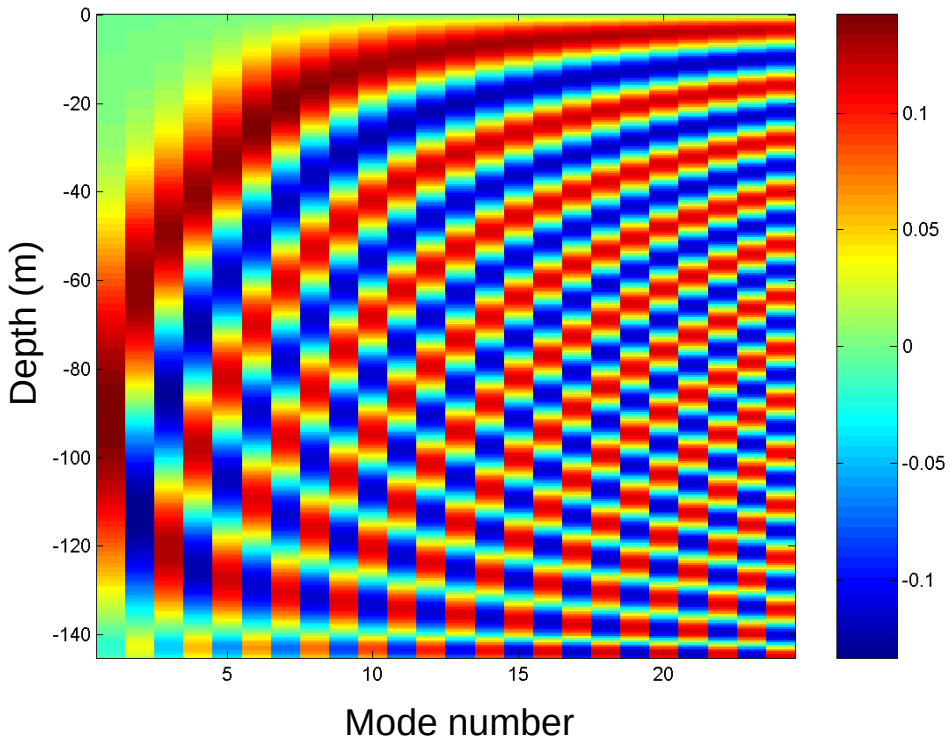
- **EOF 1 *l*=550 m, 4.9 m/s**
- **EOF 2 *l*=38 m, 2.6 m/s**



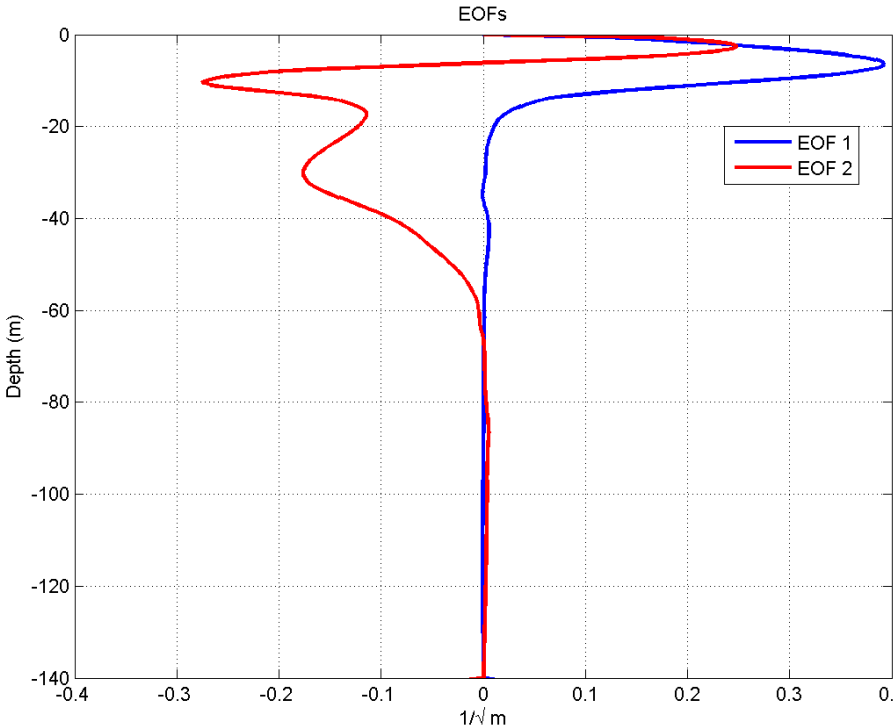


Propagation Through Summer Shallow Water Environment at 1500 Hz

Shallow water mode shapes (500 Hz)

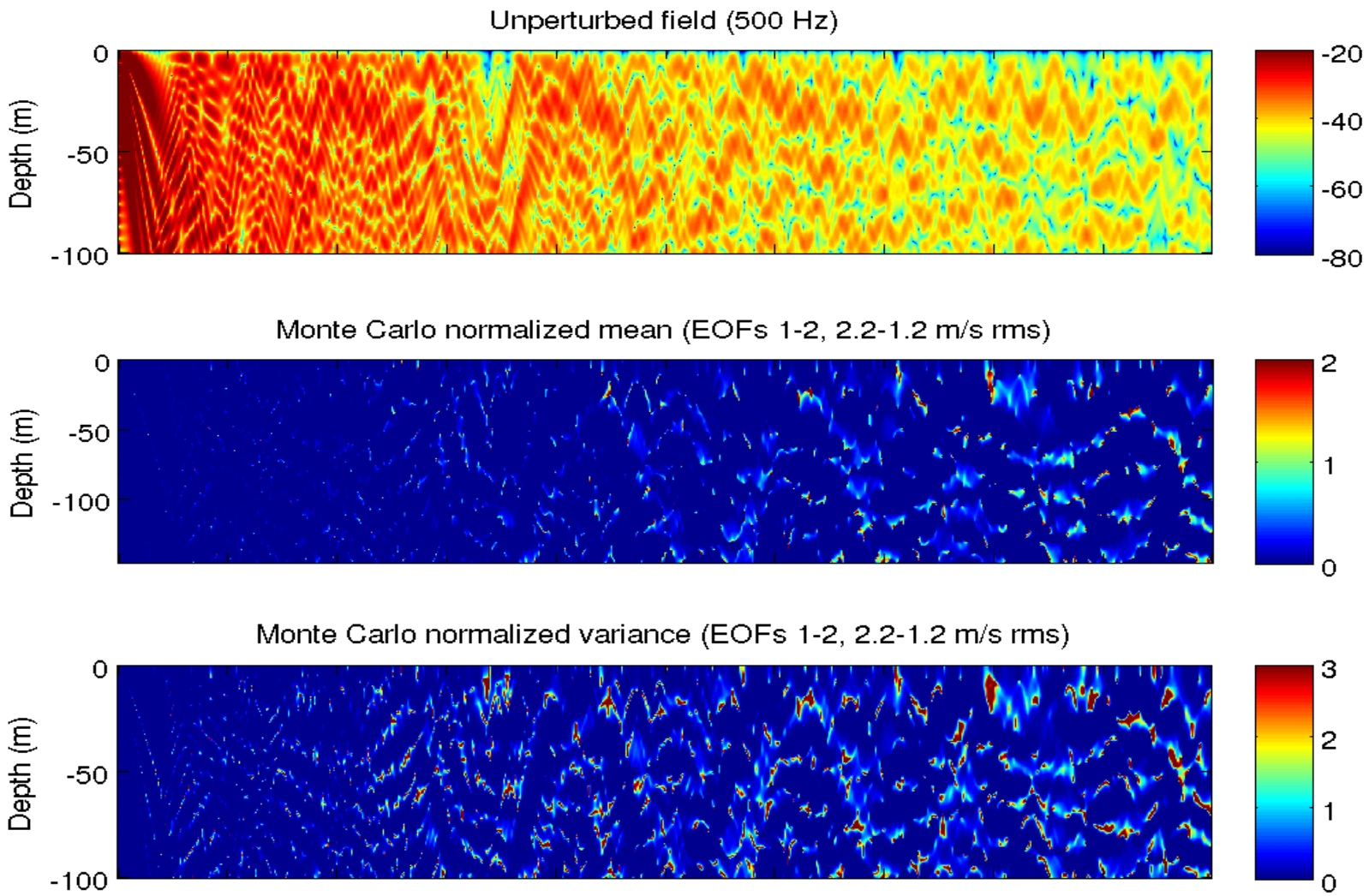


EOFs



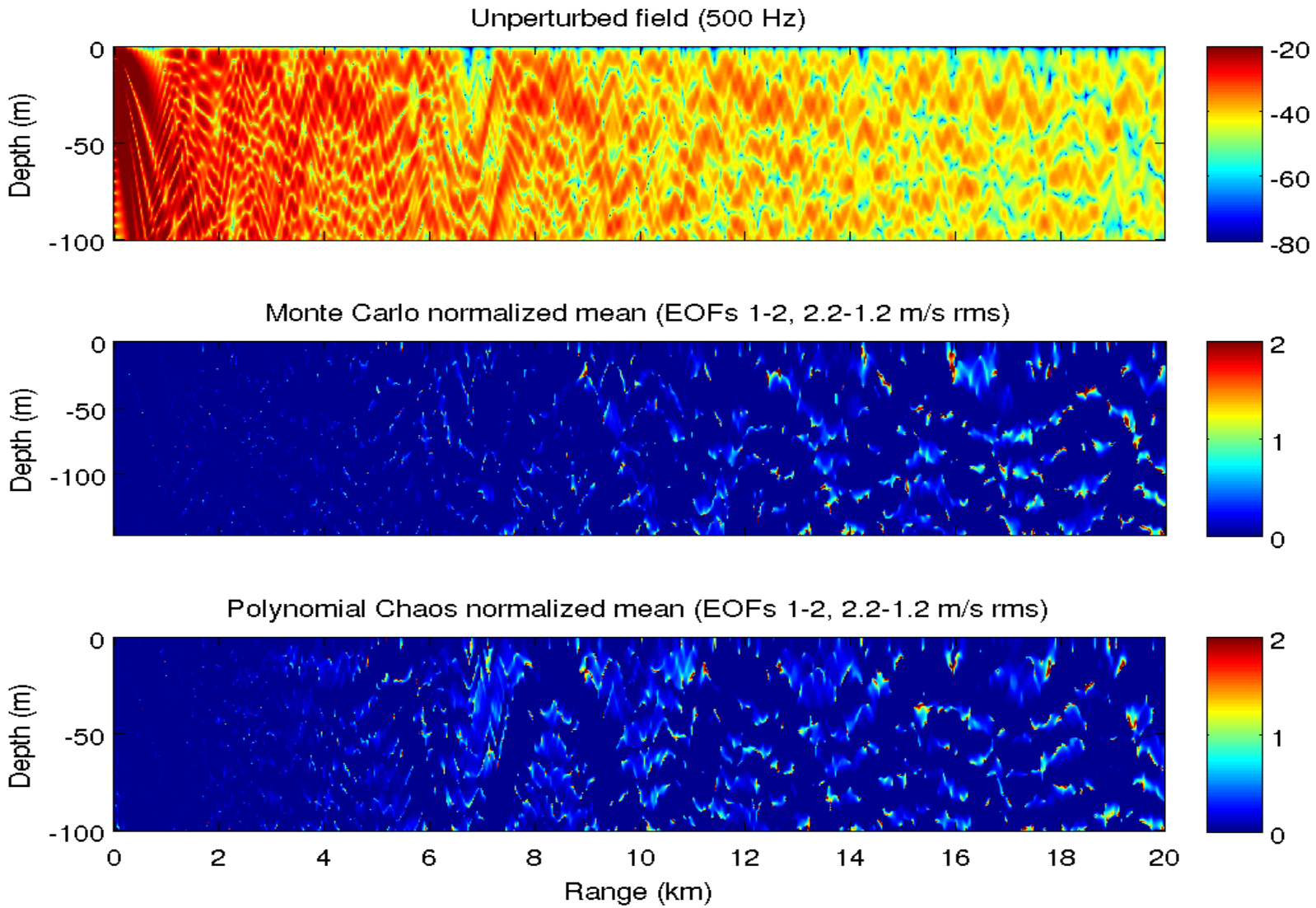
Environment to Acoustic Transfer of Uncertainty

500 Hz Yellow Shark



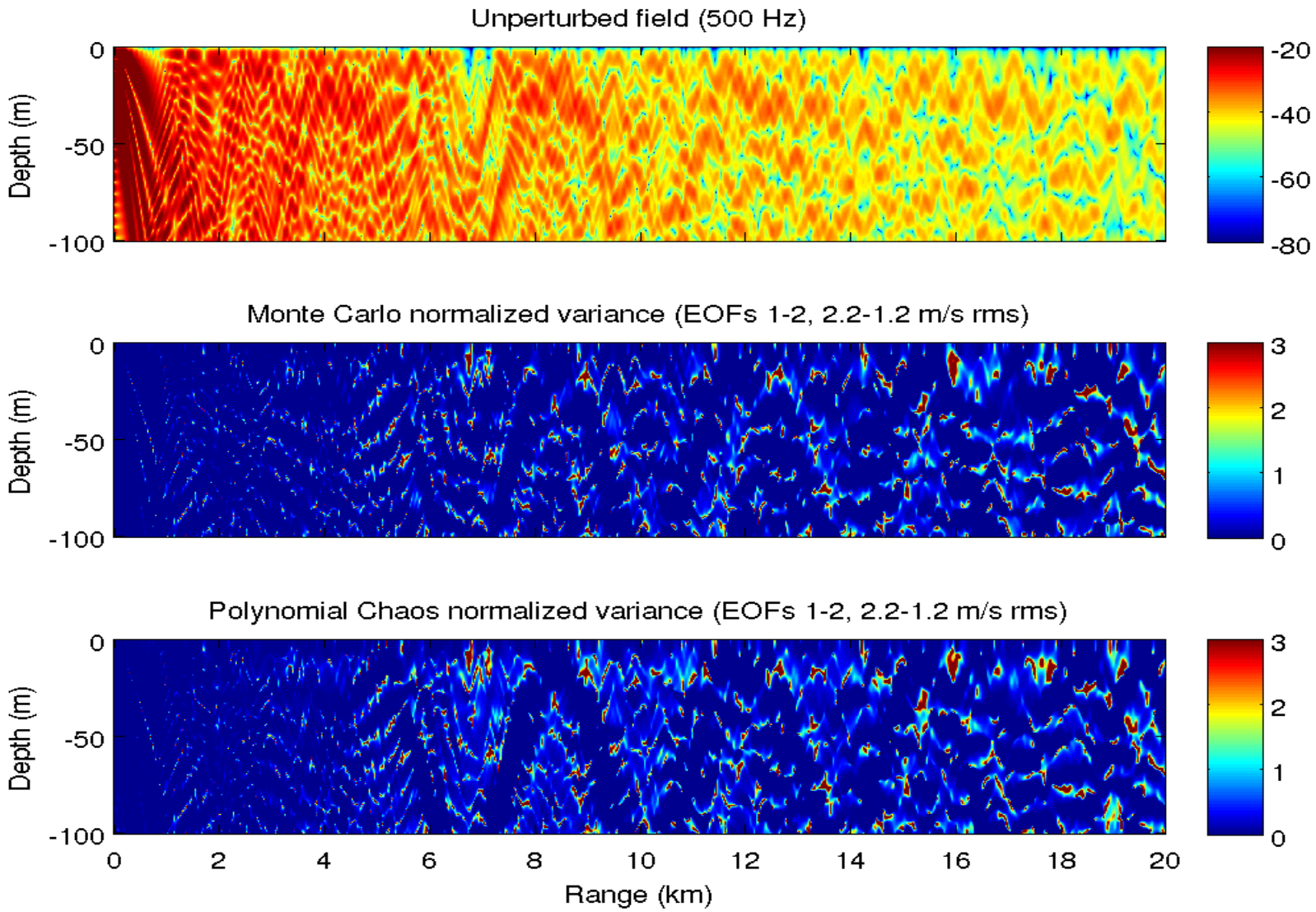
Environment to Acoustic Transfer of Uncertainty

PC vs Monte Carlo Mean at 500 Hz

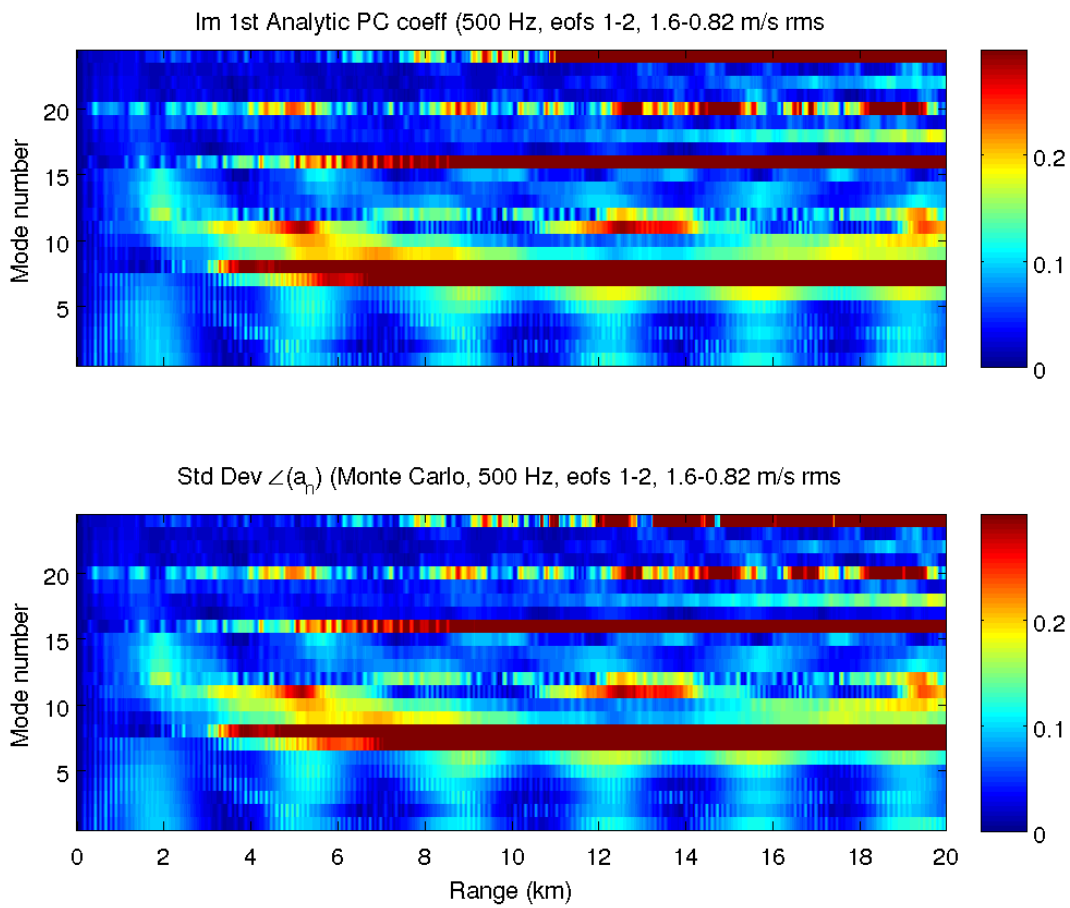


Environment to Acoustic Transfer of Uncertainty

PC vs Monte Carlo Scintillation at 500 Hz

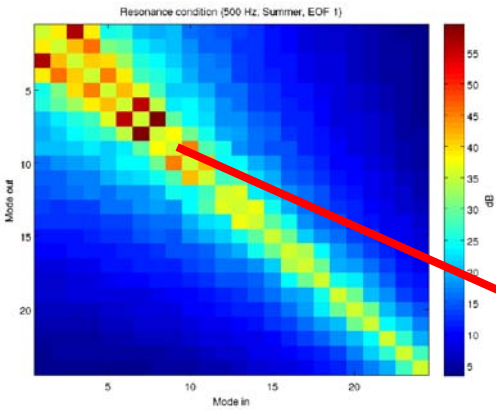


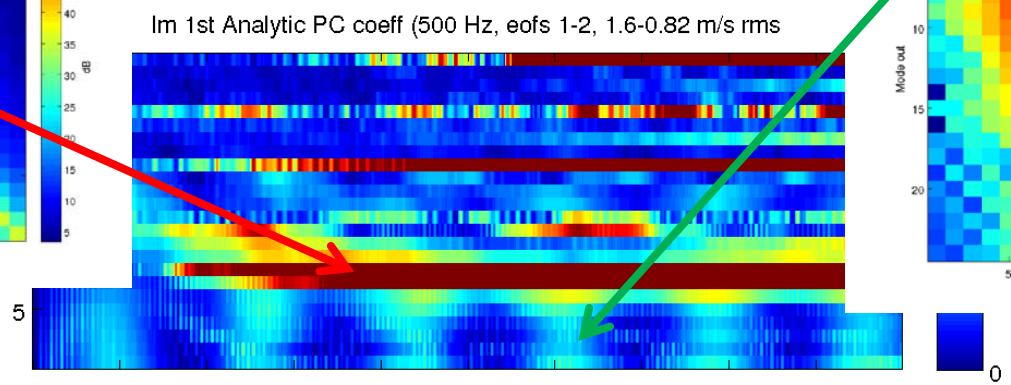
Std Dev of Angle{a_n} vs Im{PC}
500 Hz

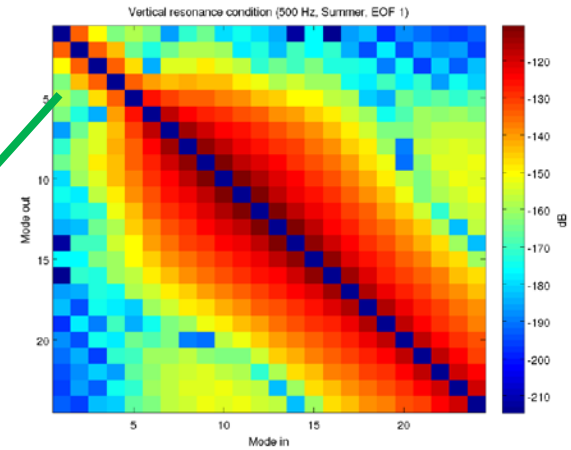


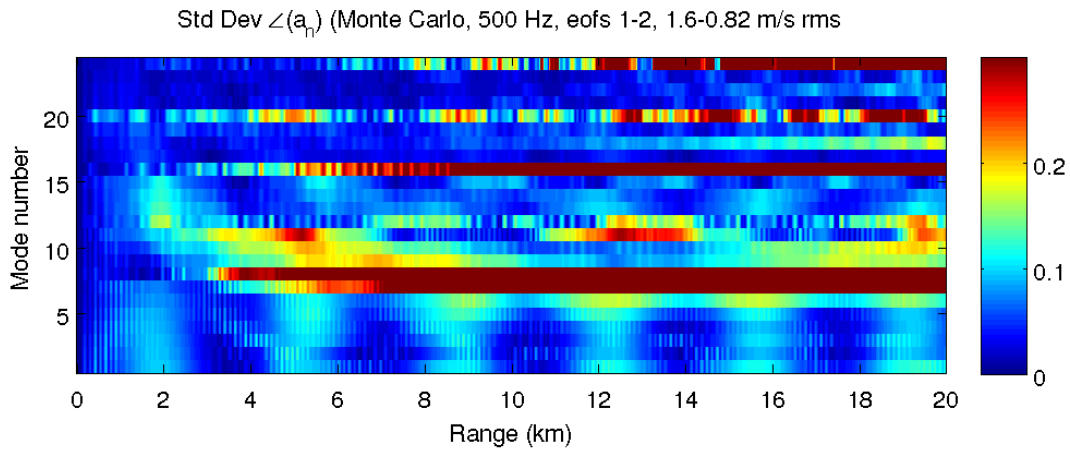
Environment to Acoustic Transfer of Uncertainty

Std Dev of Angle{a_n} vs Im{PC} 500 Hz

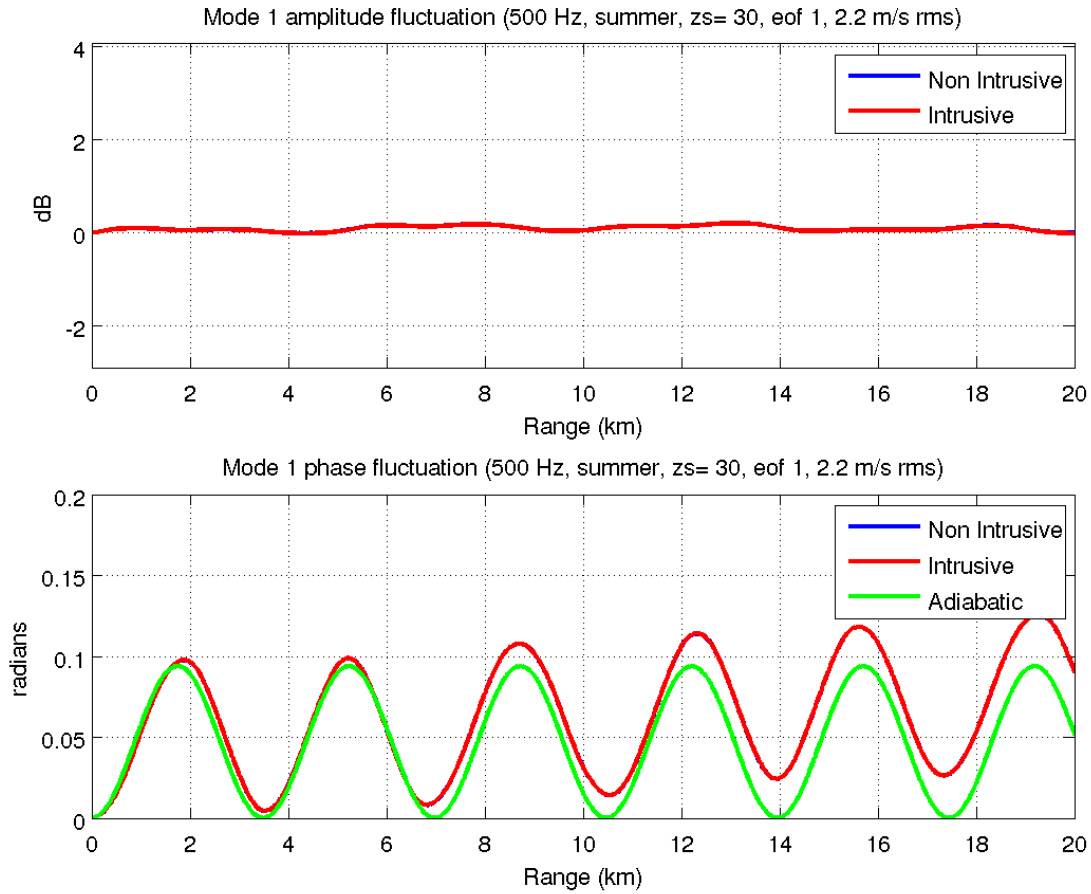




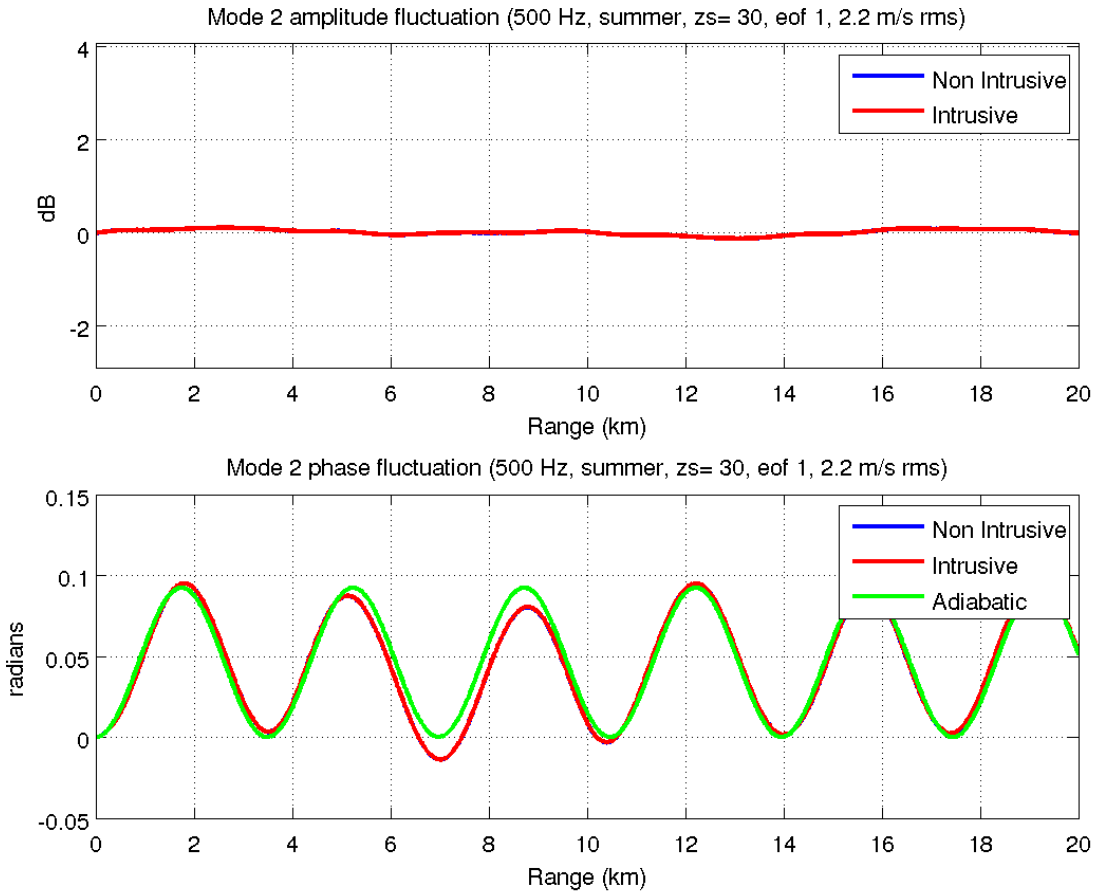




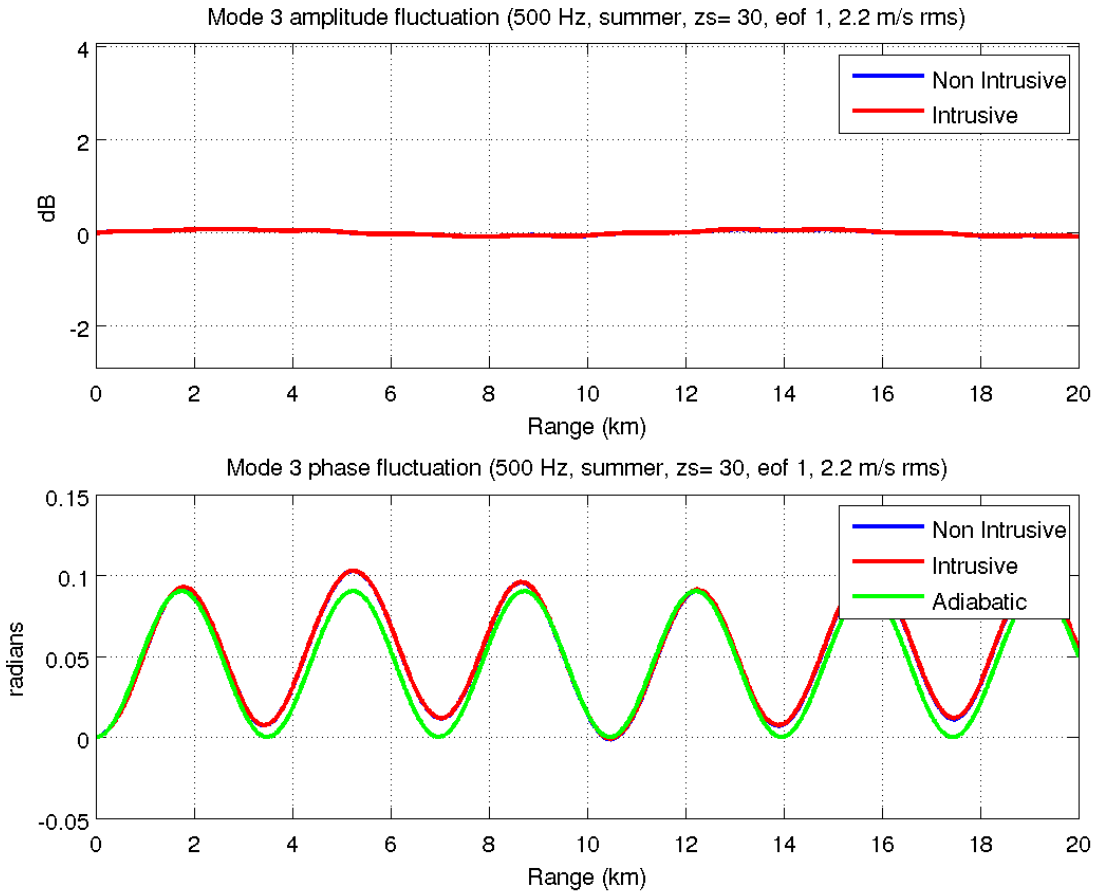
Mode 1 Adiabaticity 500 Hz



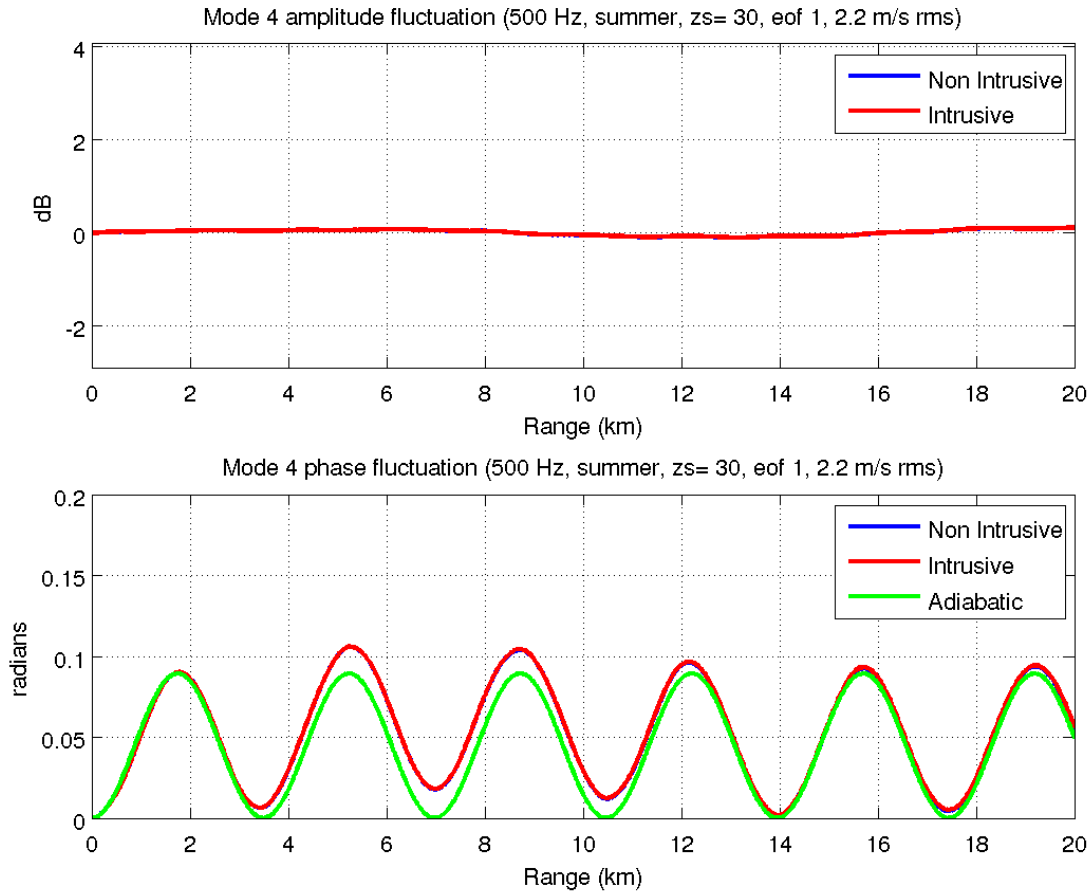
Mode 2 Adiabaticity 500 Hz



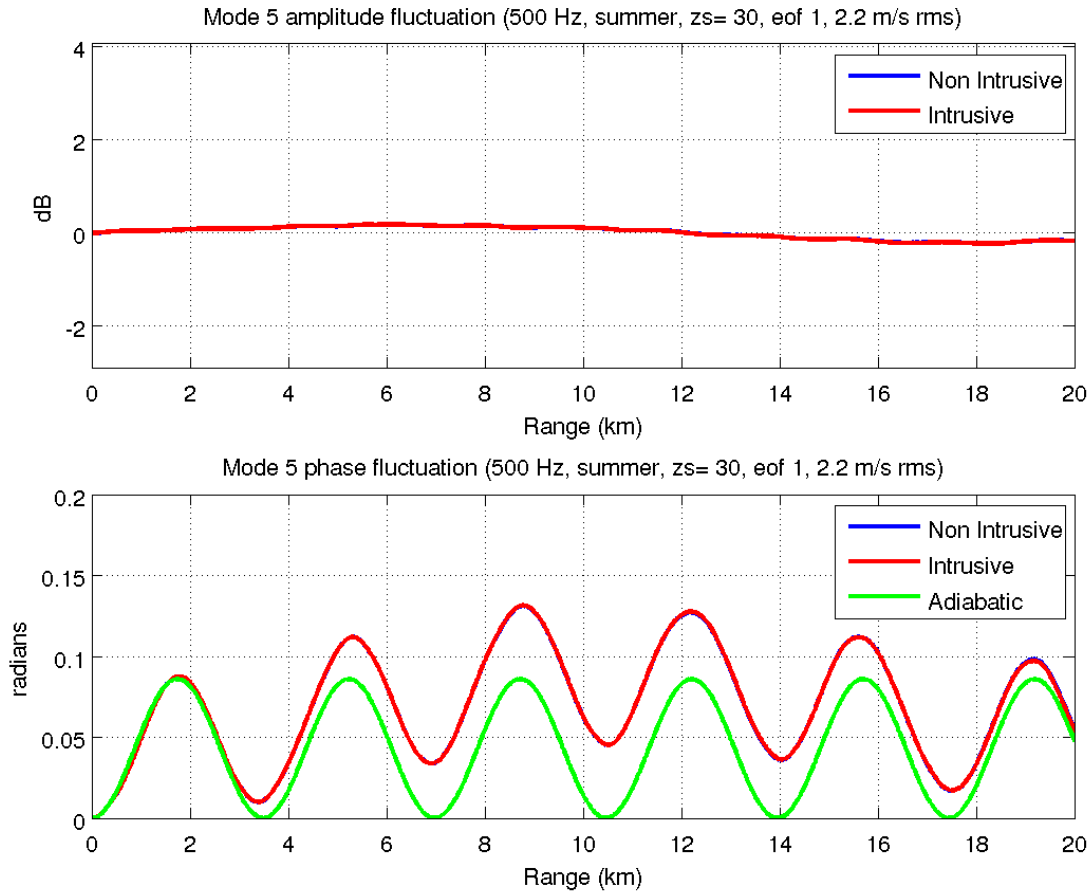
Mode 3 Adiabaticity 500 Hz



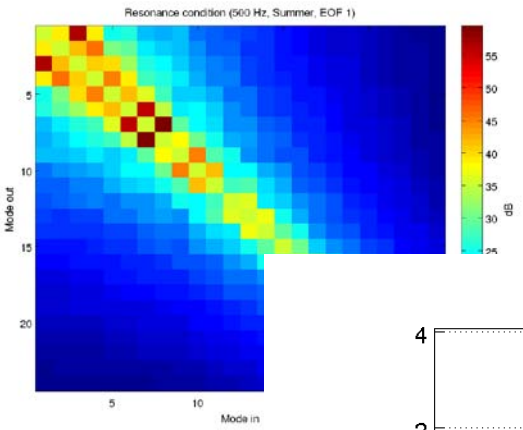
Mode 4 Adiabaticity 500 Hz

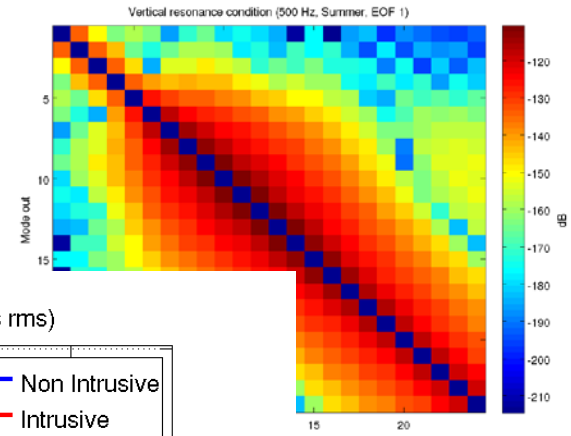


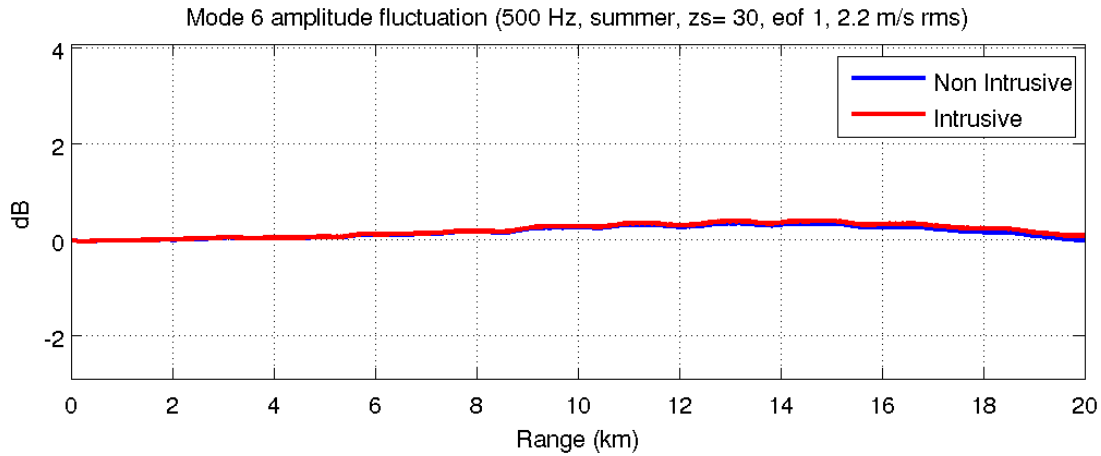
Mode 5 Adiabaticity 500 Hz

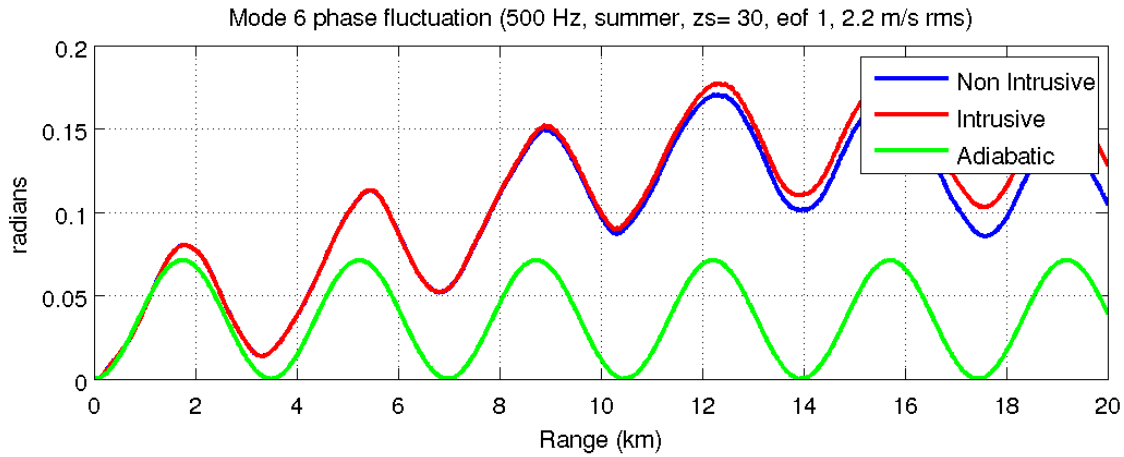


Mode 6 Adiabaticity 500 Hz

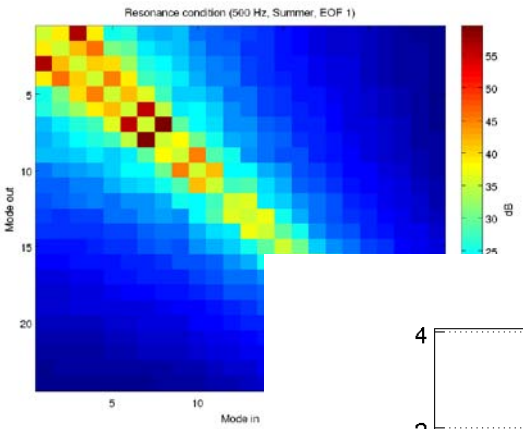


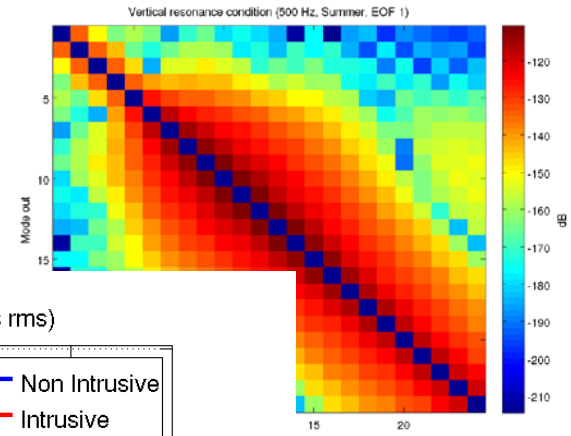


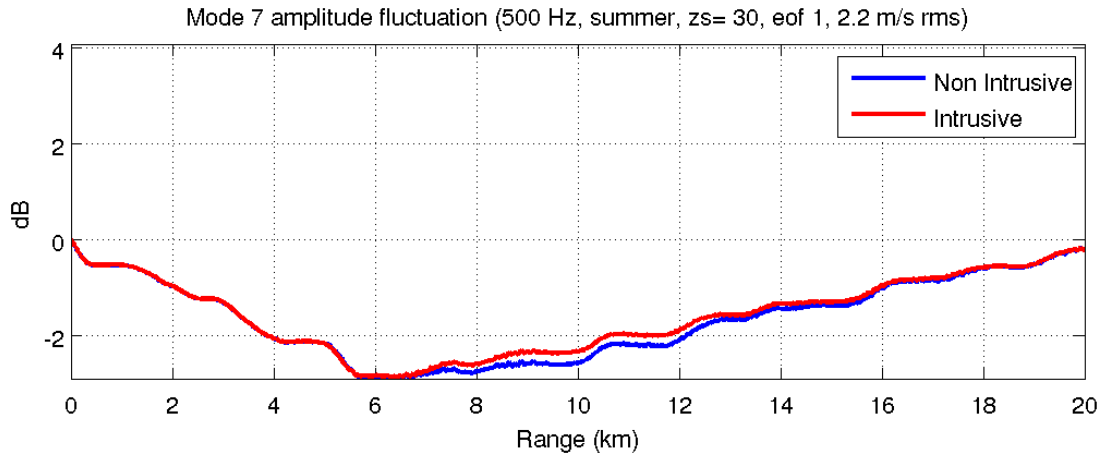


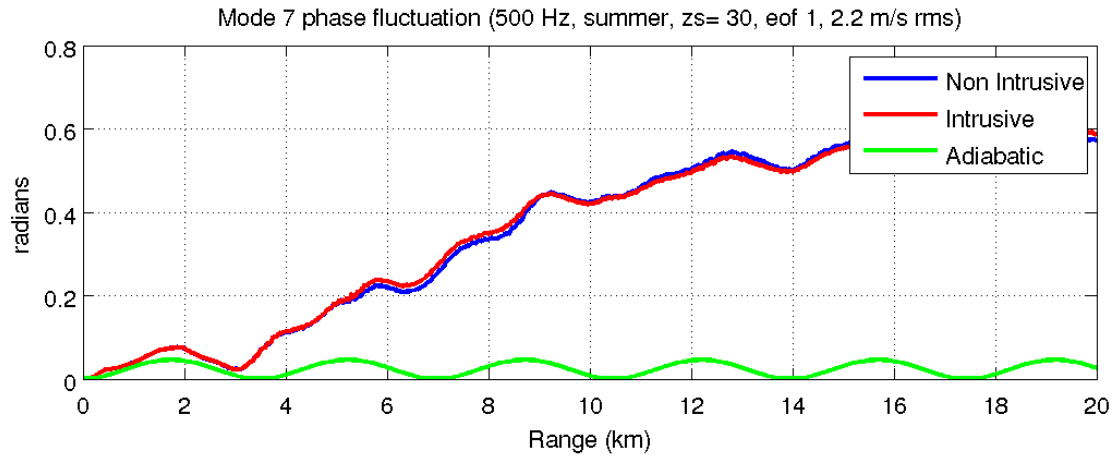


Mode 7 Adiabaticity 500 Hz

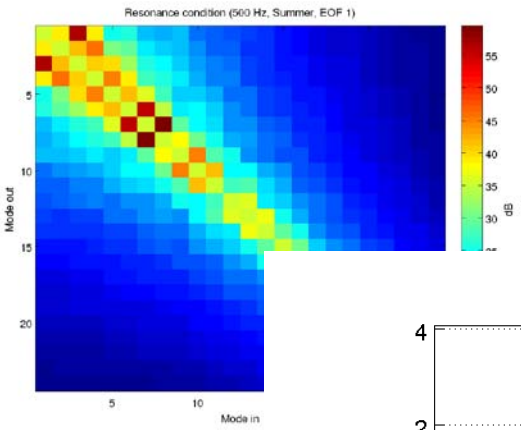


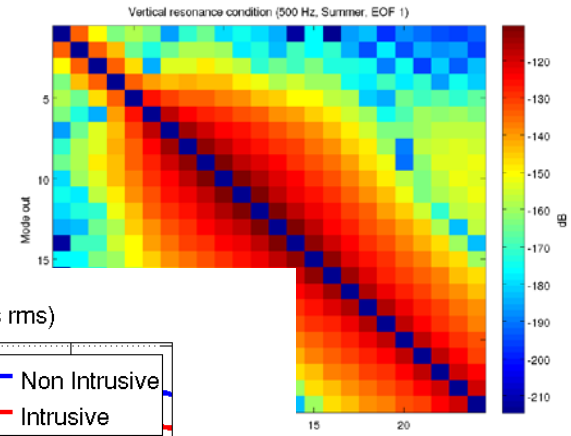


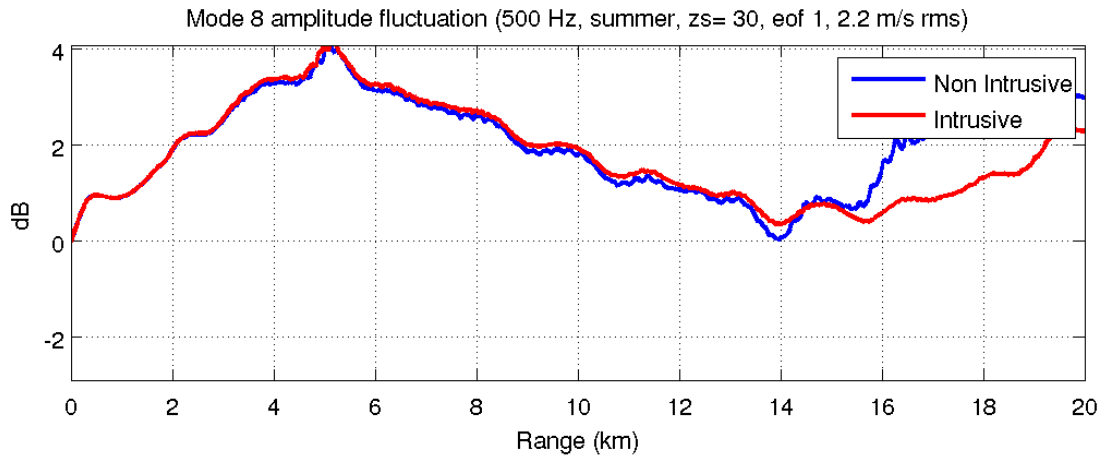


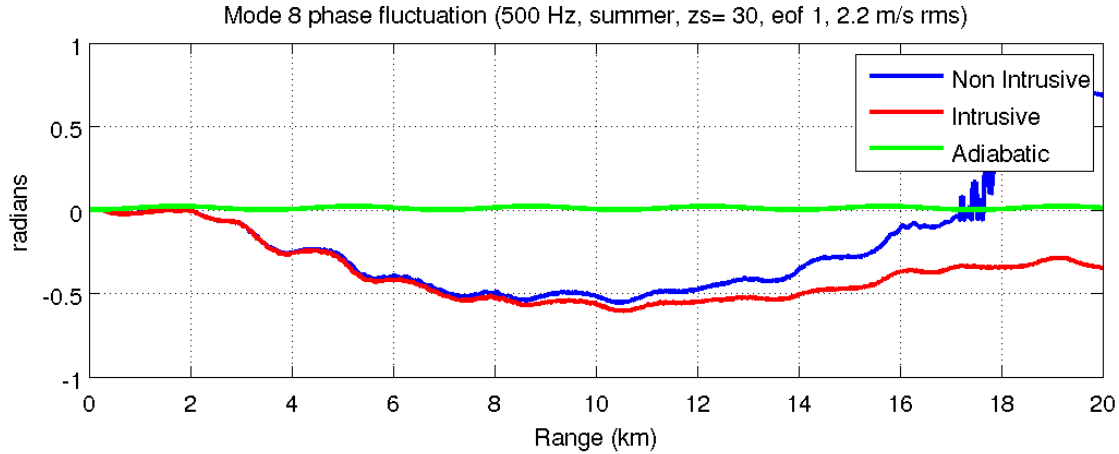


Mode 8 Adiabaticity 500 Hz

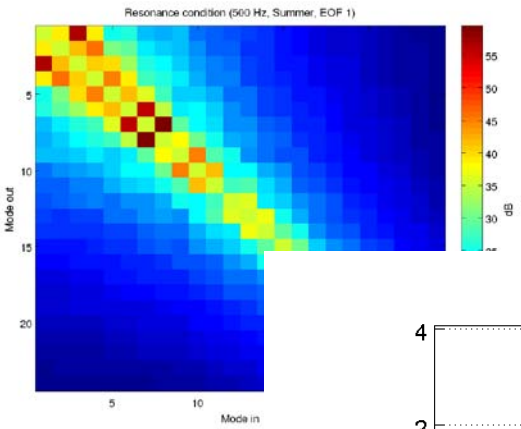


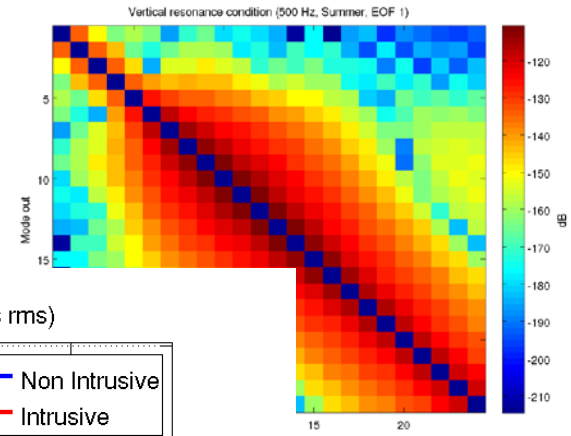


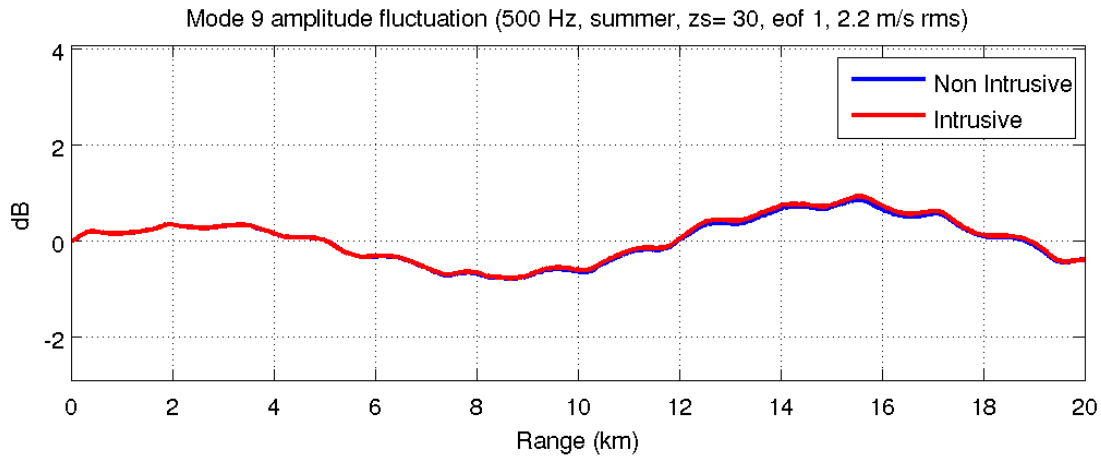


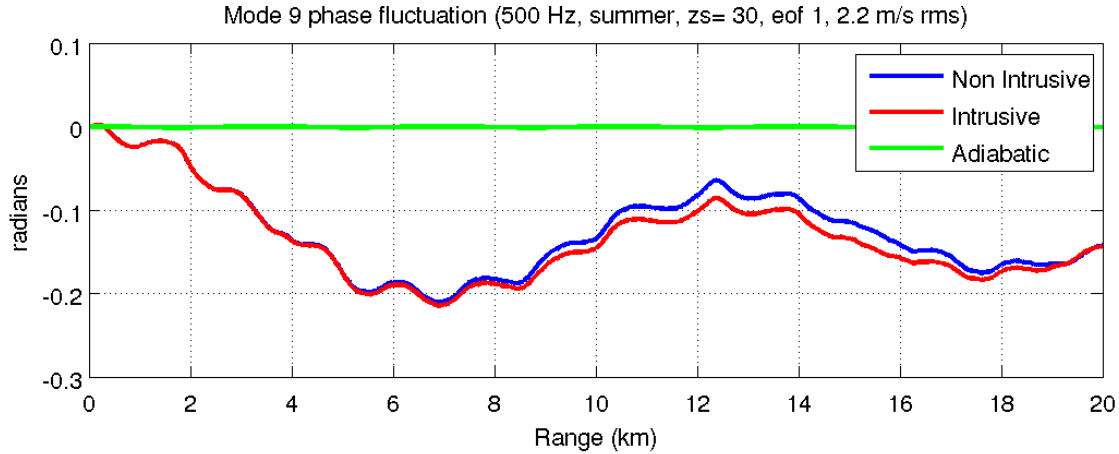


Mode 9 Adiabaticity 500 Hz

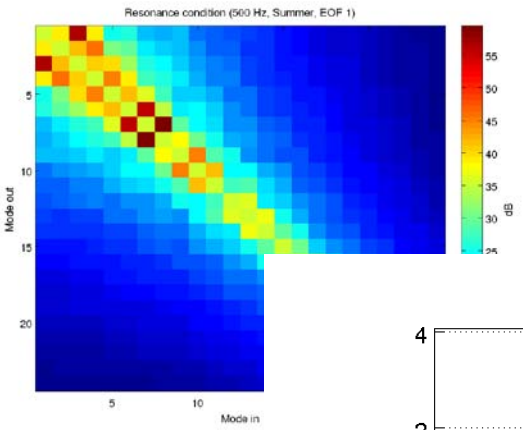


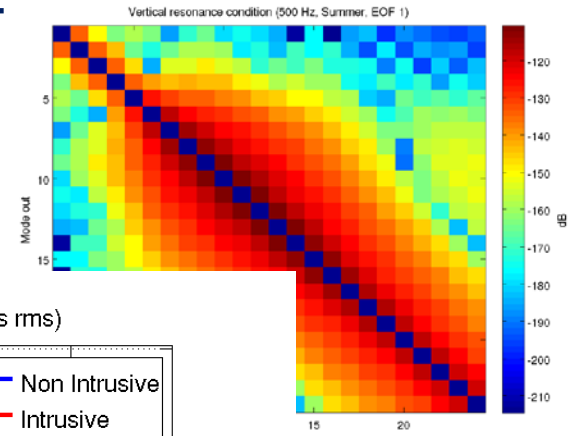


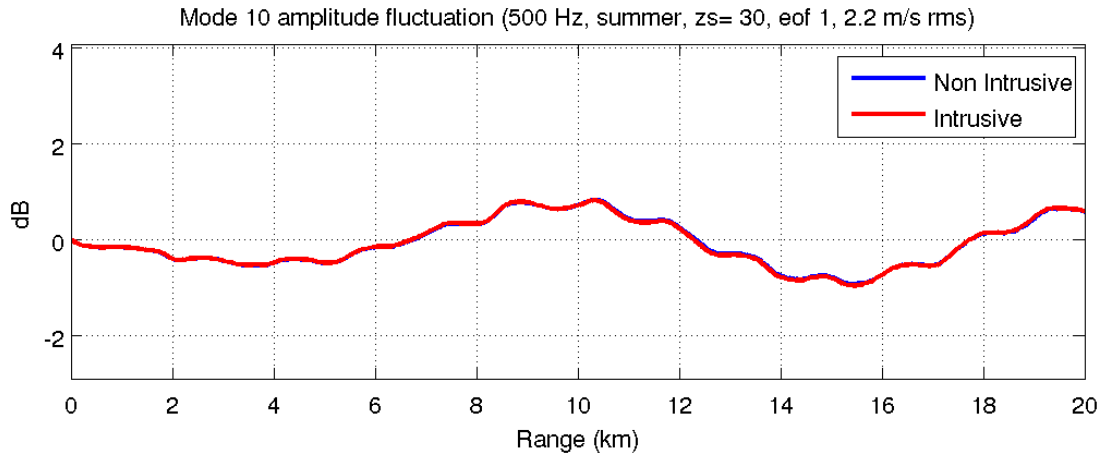


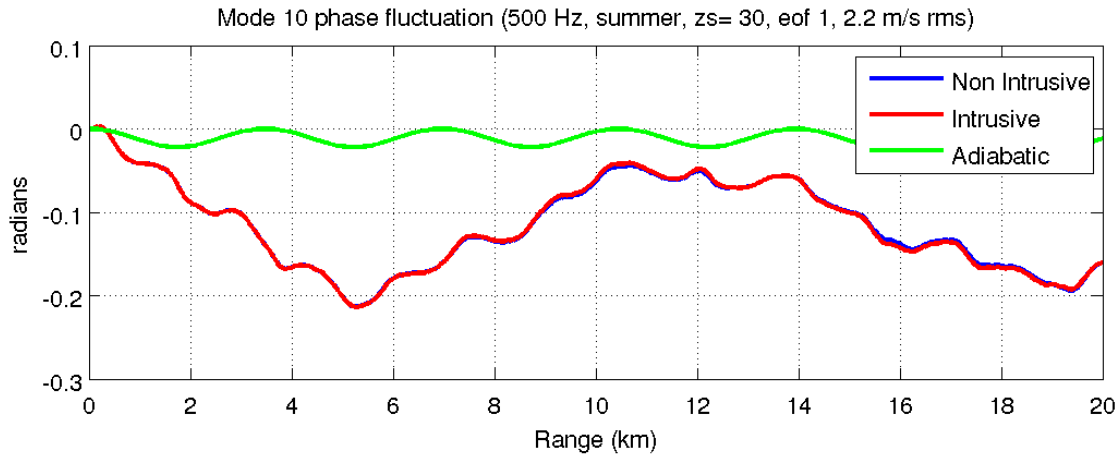


Mode 10 Adiabaticity 500 Hz

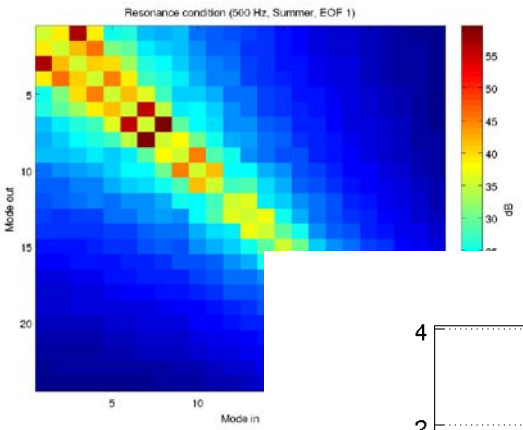


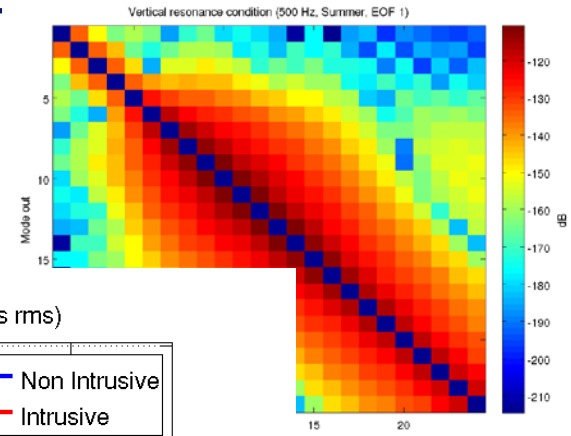


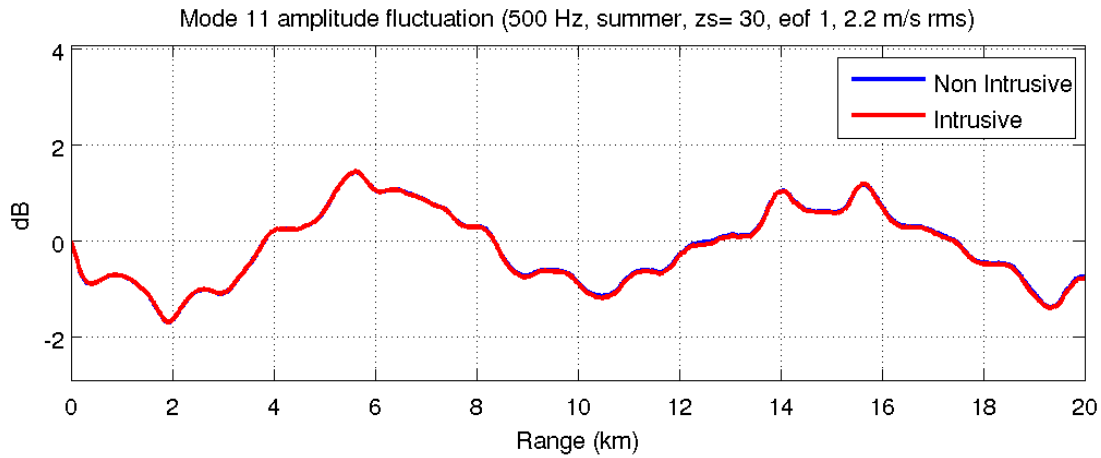


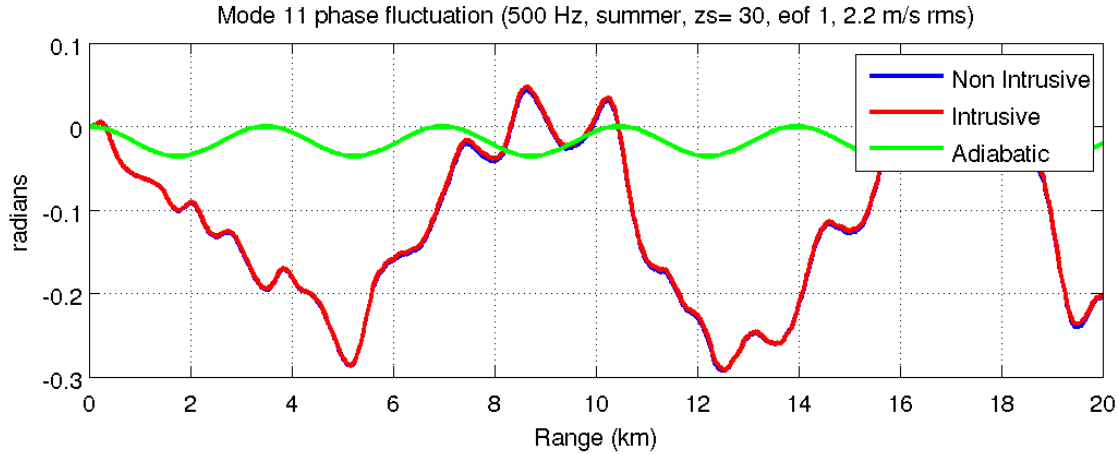


Mode 11 Adiabaticity 500 Hz









Non Adiabaticity Conclusions

- Polynomial Chaos expansion agrees closely with Monte Carlo
- Comparisons between the imaginary part of the 1st order log PC coefficient and the adiabatic approximation show good agreement for modes 1-5
- Adiabatic approximation limited to the lowest order modes which do not interact with EOF 1
- Higher order modes are strongly non-adiabatic
    - Especially modes 7 and 8 due to horizontal resonances with the 550 m horizontal correlation length scale of EOF 1
- The amplitude of modal scattering fluctuations can be as great as 4 dB

General Conclusions

- The polynomial chaos expansion of the log modal amplitude of the coupled mode equations provides a rigorous method for transfering environmental uncertainty to acoustic uncertainty
  - Various moments can be calculated
  - Probability density functions may be evaluated
- The log modal polynomial chaos expansions can also yield good estimates for when the adiabatic approximation is valid
- The PC expansion technique offers value as a diagnostic tool for evaluating the magnitude of scattering effects in coupled mode propagation
  - also for obtaining accurate solutions for the modal statistics at lower to mid frequencies and/or for slight to moderate variability
- The log modal amplitude PC equations show explicitly the contributions of horizontal resonance and the interaction between the modes and the vertical structure

Uncertainties and Risk

- Polynomial Chaos
  - High variability causes problems
    - Non-convergence of log-modal amplitude intrusive formulation
    - High N truncation of polynomial expansion for traditional intrusive formulation
  - Large numbers of degrees of freedom cause problems

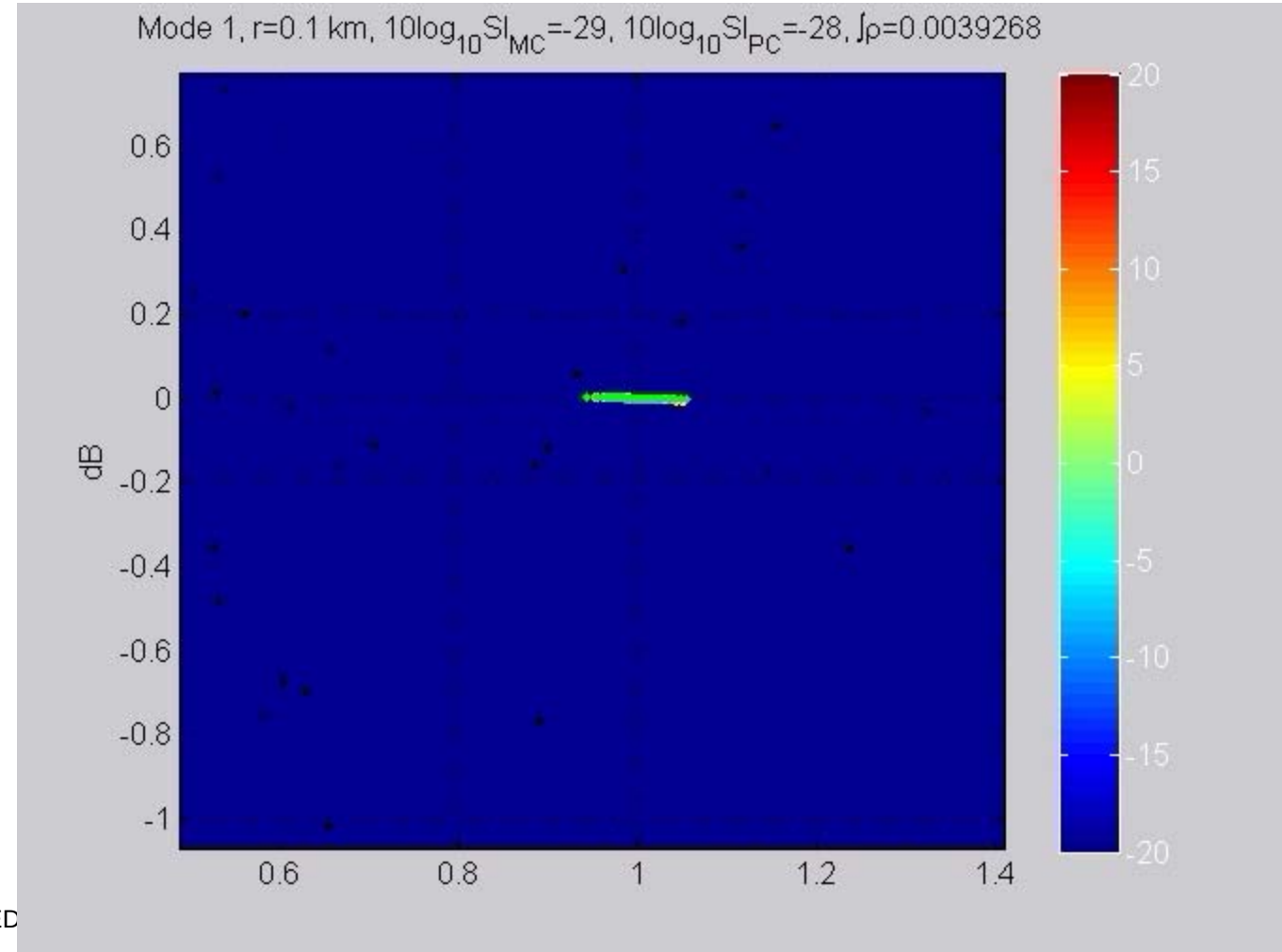
$$N_{PC} = \frac{\Gamma(n+m+1)}{\Gamma(n+1)\Gamma(m+1)}$$

  - How to handle in a probabilistic way the dumping of energy out of the waveguide at points where the horizontal slowness exceeding slowness of the bottom

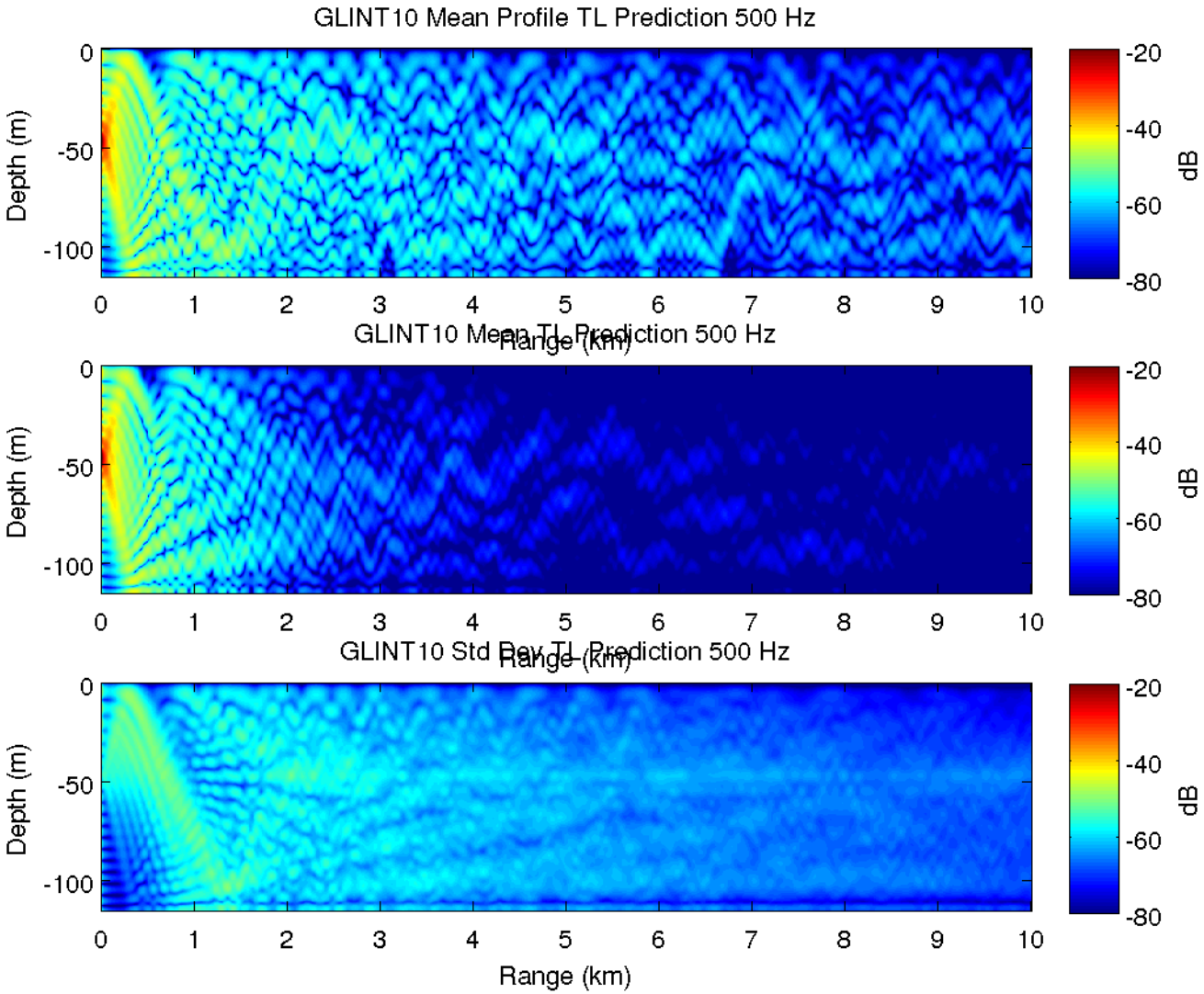
End of Talk

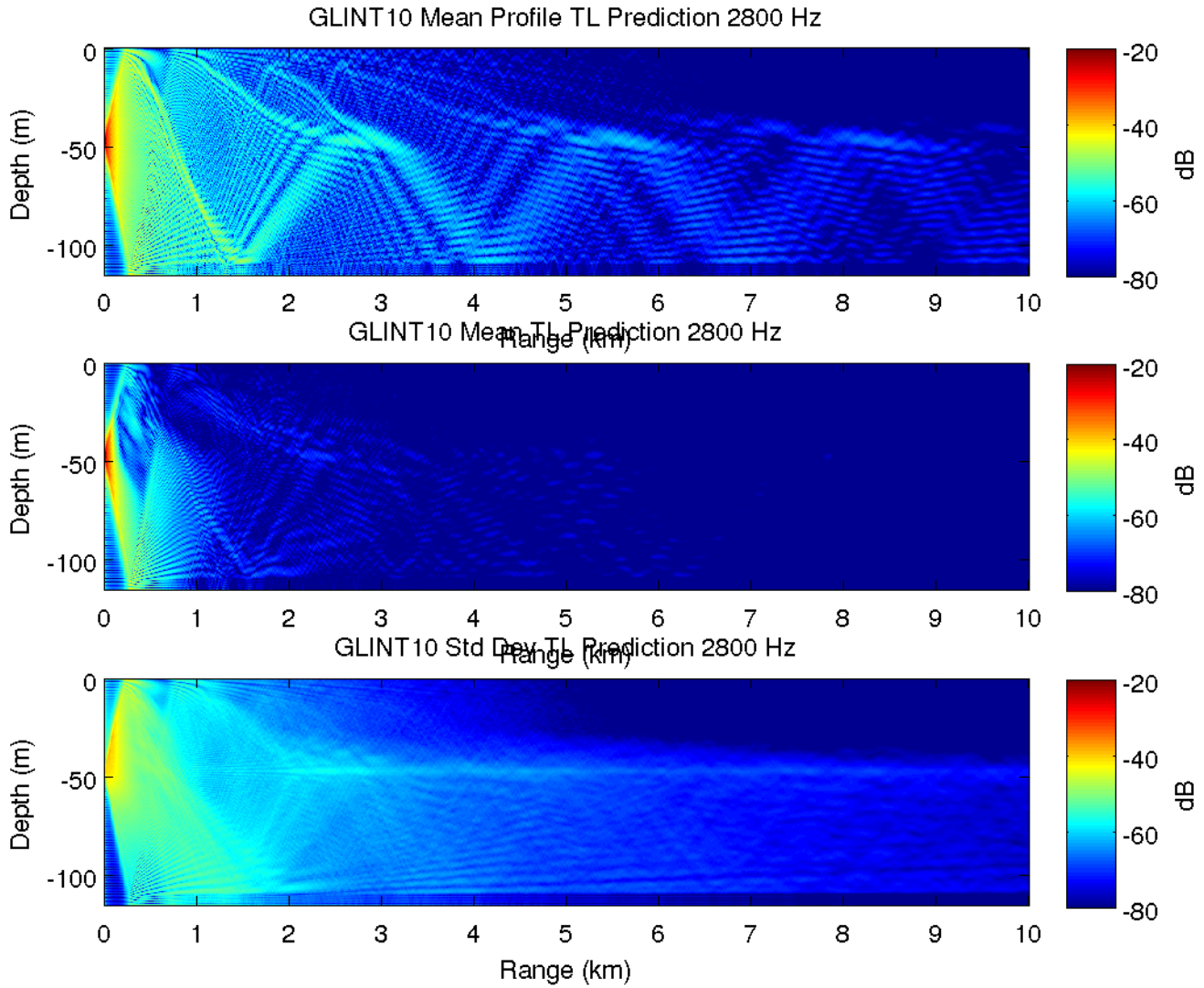
PC vs Monte Carlo *pdf*

EOFs 1-2, Multiple Modes



Resulting Acoustic Variability at 500 Hz



Environment to Acoustic Transfer of Uncertainty

Resulting Acoustic Variability at 2.8 kHz